BOUNDS ON VARIATION OF SPECTRAL SUBSPACES
UNDER *J*-SELF-ADJOINT PERTURBATIONS[*]

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ABSTRACT. Let A be a self-adjoint operator on a Hilbert space $\mathfrak{H}$. Assume that the spectrum of A consists of two disjoint components σ_0 and σ_1. Let V be a bounded operator on $\mathfrak{H}$, off-diagonal and J-self-adjoint with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ where $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are the spectral subspaces of A associated with the spectral sets σ_0 and σ_1, respectively. We find (optimal) conditions on V guaranteeing that the perturbed operator $L = A + V$ is similar to a self-adjoint operator. Moreover, we prove a number of (sharp) norm bounds on the variation of the spectral subspaces of A under the perturbation V. Some of the results obtained are reformulated in terms of the Krein space theory. As an example, the quantum harmonic oscillator under a $\mathscr{P}\mathscr{T}$-symmetric perturbation is discussed.

1. INTRODUCTION

Let A be a (possibly unbounded) self-adjoint operator on a Hilbert space $\mathfrak{H}$. Assume that V is a bounded operator on $\mathfrak{H}$. It is well known that in such a case the spectrum of the perturbed operator $L = A + V$ lies in the closed $\|V\|$-neighborhood of the spectrum of A even if V is non-self-adjoint. Thus, if the spectrum of A consists of two disjoint components σ_0 and σ_1, that is, if

$$\operatorname{spec}(A) = \sigma_0 \cup \sigma_1 \quad \text{and} \quad \operatorname{dist}(\sigma_0, \sigma_1) = d > 0, \tag{1.1}$$

then the perturbation V with a sufficiently small norm does not close the gaps between σ_0 and σ_1 in $\mathbb{C}$. This allows one to think of the corresponding disjoint spectral components σ_0' and σ_1' of the perturbed operator $L = A + V$ as a result of the perturbation of the spectral sets σ_0 and σ_1, respectively.

Assuming (1.1), by $\mathsf{E}_A(\sigma_0)$ and $\mathsf{E}_A(\sigma_1)$ we denote the spectral projections of A associated with the disjoint Borel sets σ_0 and σ_1, and by $\mathfrak{H}_0$ and $\mathfrak{H}_1$ the respective spectral subspaces, $\mathfrak{H}_0 = \operatorname{Ran} \mathsf{E}_A(\sigma_0)$ and $\mathfrak{H}_1 = \operatorname{Ran} \mathsf{E}_A(\sigma_1)$. If there is a possibility to associate with the disjoint spectral sets σ_0' and σ_1' the corresponding spectral subspaces of the perturbed (non-self-adjoint) operator $L = A + V$, we denote them by $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$. In particular, if one of the sets σ_0' and σ_1' is bounded, this can easily be done by using the Riesz projections (see, e.g. [24, Sec. III.4]).

In the present note we are mainly concerned with bounded perturbations V that possess the property

$$V^* = JVJ, \tag{1.2}$$

where J is a self-adjoint involution on $\mathfrak{H}$ given by

$$J = \mathsf{E}_A(\sigma_0) - \mathsf{E}_A(\sigma_1). \tag{1.3}$$

Operators V with the property (1.2) are called J-self-adjoint.

A bounded perturbation V is called diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ if it commutes with the involution J, $VJ = JV$. If V anticommutes with J, i.e. $VJ = -JV$, then V is said to be off-diagonal. Clearly, any bounded V can be represented as the sum $V = V_{\mathrm{diag}} + V_{\mathrm{off}}$ of the diagonal, V_{diag}, and off-diagonal, V_{off}, terms. The spectral subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ remain invariant under $A + V_{\mathrm{diag}}$ while adding a non-zero V_{off} does break the invariance of $\mathfrak{H}_0$ and $\mathfrak{H}_1$. Thus, the core of the perturbation theory for spectral subspaces is in the study of their variation under off-diagonal perturbations (cf. [25]). This is the reason why we add to the hypothesis (1.2) another basic assumption, namely that all the perturbations V involved are off-diagonal with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$.

We recall that if an off-diagonal perturbation V is self-adjoint in the usual sense, that is, $V^* = V$, then the condition

$$\|V\| < \frac{d}{2} \tag{1.4}$$

ensuring the existence of gaps between the perturbed spectral sets σ_0' and σ_1' may be essentially relaxed. Generically, if no assumptions on the mutual position of the initial spectral sets σ_0 and σ_1 are made except (1.1), the sets σ_0' and σ_1' remain disjoint for any off-diagonal self-adjoint V satisfying the bound $\|V\| < \frac{\sqrt{3}}{2}d$ (see [27, Theorem 1 (i)]). If, in addition to (1.1), it is known that one of the sets σ_0 and σ_1 lies in a finite gap of the other set then this bound may be relaxed further: for the perturbed sets σ_0' and σ_1' to be disjoint it only suffices to require that $\|V\| < \sqrt{2}d$ (see [27, Theorem 2 (i)]; cf. [26, Remark 3.3]). Finally, if the sets σ_0 and σ_1 are subordinated, say $\sup \sigma_0 < \inf \sigma_1$, then no requirements on $\|V\|$ are needed at all: the interval $(\sup \sigma_0, \inf \sigma_1)$ belongs to the resolvent set of the perturbed operator $L = A + V$ for any bounded off-diagonal self-adjoint V (see [2, 17, 33]; cf. [28]) and even for some off-diagonal unbounded symmetric V (see [41, Theorem 1]). It is easily seen from Example 5.5 below that in the case of J-self-adjoint off-diagonal perturbations the condition (1.4) ensuring the disjointness of the perturbed spectral sets σ_0' and σ_1' can be relaxed for none of the above dispositions of the initial spectral sets σ_0 and σ_1.

Assuming that V is a bounded J-self-adjoint off-diagonal perturbation of the (possibly unbounded) self-adjoint operator A we address the following questions:

(i) Does the spectrum of the perturbed operator $L = A + V$ remain real under conditions (1.1) and (1.4)?

(ii) If yes, is it then true that L is similar to a self-adjoint operator?

(iii) What are the (sharp) bounds on variation of the spectral subspaces associated with the spectral sets σ_0 and σ_1 as well as on the variation of these sets themselves?

In our answers to the above questions we distinguish two cases:

(G) the generic case where no assumptions on the mutual positions of the spectral sets σ_0 and σ_1 are done except for the disjointness assumption (1.1);

(S) the particular case where the sets σ_0 and σ_1 are either subordinated, e.g. $\sup \sigma_0 < \inf \sigma_1$, or one of these sets lies in a finite gap of the other set, say σ_0 lies in a finite gap of σ_1.

We have to underline that this distinction is quite different from the one that arises when the perturbations V are self-adjoint in the usual sense: the case (S) now combines the two spectral dispositions that should be treated separately if V were self-adjoint (see [7, 17, 27, 41]).

Our answers to the questions (i) and (ii) are complete and positive in the case (S). In this case the spectrum of the perturbed operator $L = A + V$ does remain real for any off-diagonal J-self-adjoint V satisfying the bound $\|V\| \leq d/2$. Moreover, the operator L turns out to be similar to a

self-adjoint operator whenever the strict inequality (1.4) holds. These results combined in Theorem 5.8 (ii) below (see also Remark 5.13) represent an extension of similar results previously known due to [1] and [37] for the spectral dispositions with subordinated σ_0 and σ_1.

By using the results of [32, 49], we give a positive answer to the question (i) also in the generic case (G) provided that the unperturbed operator A is bounded (see Theorem 5.12). For A unbounded, we prove that in case (G) the spectrum of $L = A + V$ for sure is purely real if V satisfies a stronger bound $\|V\| \leq d/\pi$. The strict bound $\|V\| < d/\pi$ guarantees, in addition, that L is similar to a self-adjoint operator $\big($see Theorem 5.8 (i)$\big)$. The question whether this is true for $d/\pi \leq \|V\| < d/2$ remains an open problem.

We answer the question (iii) by using the concept of the operator angle between two subspaces (for discussion of this notion and references see, e.g., [25]). Recall that if $\mathfrak{M}$ and $\mathfrak{N}$ are subspaces of a Hilbert space, the operator angle $\Theta(\mathfrak{M}, \mathfrak{N})$ between $\mathfrak{M}$ and $\mathfrak{N}$ measured relative to the subspace $\mathfrak{M}$ is introduced by the following formula [26]:

$$\Theta(\mathfrak{M}, \mathfrak{N}) = \arcsin \sqrt{I_{\mathfrak{M}} - P_{\mathfrak{M}} P_{\mathfrak{N}}}\big|_{\mathfrak{M}}, \tag{1.5}$$

where $I_{\mathfrak{M}}$ denotes the identity operator on $\mathfrak{M}$ and $P_{\mathfrak{M}}$ and $P_{\mathfrak{N}}$ stand for the orthogonal projections onto $\mathfrak{M}$ and $\mathfrak{N}$, respectively.

Set

$$\delta = \begin{cases} 2d/\pi, & \text{case (G)}, \\ d, & \text{case (S)}, \end{cases} \tag{1.6}$$

and assume that $\|V\| < \delta/2$. Since in both the cases (G) and (S) under this assumption we have got the positive answer to the question (ii), one can easily identify the spectral subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ of L associated with the corresponding perturbed spectral sets σ_0' and σ_1' (cf. Lemma 5.6). Let $\Theta_j = \Theta(\mathfrak{H}_j, \mathfrak{H}_j')$, $j = 0, 1$, be the operator angle between the unperturbed spectral subspace $\mathfrak{H}_j$ and the perturbed one, $\mathfrak{H}_j'$. Our main result (presented in Theorem 5.8) regarding the operator angles Θ_0 and Θ_1 is that under condition $\|V\| < \delta/2$ the following bound holds:

$$\tan \Theta_j \leq \tanh \left(\frac{1}{2} \operatorname{arctanh} \frac{2\|V\|}{\delta} \right), \quad j = 0, 1, \tag{1.7}$$

which means, in particular, that $\Theta_j < \frac{\pi}{4}$, $j = 0, 1$. Theorem 5.8 also gives the bounds on location of the perturbed spectral sets σ_0' and σ_1' $\big($see formulas (5.19)$\big)$.

In the case (S) the bounds on σ_0' and σ_1' as well as the bounds (1.7) are optimal (see Remark 5.10). Inequalities (1.7) resemble the sharp norm estimate for the operator angle between perturbed and unperturbed spectral subspaces from the celebrated Davis-Kahan $\tan 2\Theta$ Theorem (see [17], p. 11; cf. [28, Theorem 2.4] and [41, Theorem 1]). Recall that the latter theorem serves for the case where the unperturbed spectral subsets σ_0 and σ_1 are subordinated and the off-diagonal perturbation V is self-adjoint. The difference is that the usual tangent of the Davis-Kahan $\tan 2\Theta$ Theorem is replaced on the right-hand side of (1.7) by the hyperbolic one. Another distinction is that the bound (1.7) holds not only for the subordinated spectral sets σ_0 and σ_1 but also for the disposition where one of these sets lies in a finite gap of the other set and thus σ_0 and σ_1 are not subordinated.

The results obtained are of particular interest for the theory of operators on Krein spaces [9]. The reason for this is that introducing an indefinite inner product $[x, y] = (Jx, y)$, $x, y \in \mathfrak{H}$, instead of the initial inner product $(\cdot, \cdot)$, turns $\mathfrak{H}$ into a Krein space. The operators V and $L = A + V$ being J-self-adjoint on $\mathfrak{H}$ appear to be self-adjoint operators on the newly introduced Krein space $\mathfrak{K}$. Under the condition $\|V\| < \delta/2$ in both cases (G) and (S) we establish that the perturbed spectral subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are mutually orthogonal with respect to the inner product $[\cdot, \cdot]$. Moreover,

these subspaces are maximal uniformly positive and maximal uniformly negative, respectively (see Remark 5.11). The restrictions of L onto $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are $\mathfrak{K}$-unitary equivalent to self-adjoint operators on $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. This extends similar results previously known from [1] and [37] for the case where the spectral sets σ_0 and σ_1 are subordinated.

Another motivation for the present paper is in the spectral analysis of non-self-adjoint Schrödinger operators that involve the so-called $\mathscr{PT}$-symmetric potentials. Starting from the pioneering works [11, 12], these potentials attracted considerable attention because of their property to produce, in some cases, purely real spectra (see, e.g., [3, 4, 10, 29, 39, 50]). The local $\mathscr{PT}$-symmetric potentials appear to be J-self-adjoint with respect to the space parity operator $\mathscr{P}$ (see, e.g., [32, 39]), allowing for an embedding the problem into the context of the spectral theory for J-self-adjoint perturbations (this also means that the $\mathscr{PT}$-symmetric perturbations may be studied within the framework of the Krein space theory [4, 32, 48]).

The main tool we use in our analysis is a reduction of the problems (i)–(iii) to the study of the operator Riccati equation

$$KA_0 + A_1K + KBK = -B^*$$

associated with the representation of the perturbed operator $L = A + V$ in the 2×2 block matrix form

$$L = \begin{pmatrix} A_0 & B \\ -B^* & A_1 \end{pmatrix},$$

where $A_0 = A\big|_{\mathfrak{H}_0}$, $A_1 = A\big|_{\mathfrak{H}_1}$, and $B = V\big|_{\mathfrak{H}_1}$. Assuming (1.6), we prove that the Riccati equation has a bounded solution K for any B such that $\|B\| < \delta/2$. The key statement is that the perturbed spectral subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are the graphs of the operators K and K^*, respectively, which then allows us to derive the bounds (1.7).

The plan of the paper is as follows. In Section 2 we give necessary definitions and present some basic results on the operator Riccati equations associated with a class of unbounded non-self-adjoint 2×2 block operator matrices. Section 3 is devoted to the related Sylvester equations. In Section 4 we prove a number of existence and uniqueness results for the operator Riccati equations. In Section 5 we consider J-self-adjoint perturbations and find conditions on their norm guaranteeing the reality of the resulting spectrum. In this section we also prove the bound (1.7) on the variation of the spectral subspaces and discuss the embedding of the problem into the context of the Krein space theory. Finally, in Section 6 we apply some of the results obtained to a quantum-mechanical Hamiltonian describing the harmonic oscillator under a $\mathscr{PT}$-symmetric perturbation.

We conclude the introduction with the description of some more notations that are used throughout the paper. By a subspace we always understand a closed linear subset of a Hilbert space. The identity operator on a subspace (or on the whole Hilbert space) $\mathfrak{M}$ is denoted by $I_{\mathfrak{M}}$. If no confusion arises, the index $\mathfrak{M}$ may be omitted in this notation. The Banach space of bounded linear operators from a Hilbert space $\mathfrak{M}$ to a Hilbert space $\mathfrak{N}$ is denoted by $\mathscr{B}(\mathfrak{M}, \mathfrak{N})$. For $\mathscr{B}(\mathfrak{M}, \mathfrak{M})$ we use a shortened notation $\mathscr{B}(\mathfrak{M})$. By $\mathfrak{M} \oplus \mathfrak{N}$ we will understand the orthogonal sum of two Hilbert spaces (or orthogonal subspaces) $\mathfrak{M}$ and $\mathfrak{N}$. By $\mathscr{O}_r(\mathfrak{M}, \mathfrak{N})$, $0 \leq r < \infty$, we denote the closed ball in $\mathscr{B}(\mathfrak{M}, \mathfrak{N})$, having radius r and being centered at zero, that is,

$$\mathscr{O}_r(\mathfrak{M}, \mathfrak{N}) = \{K \in \mathscr{B}(\mathfrak{M}, \mathfrak{N}) \mid \|K\| \leq r\}.$$

If it so happens that $r = +\infty$, by $\mathscr{O}_\infty(\mathfrak{M}, \mathfrak{N})$ we will understand the whole space $\mathscr{B}(\mathfrak{M}, \mathfrak{N})$. The notation $\mathrm{conv}(\sigma)$ is used for the convex hull of a Borel set $\sigma \subset \mathbb{R}$. By $O_r(\Omega)$, $r \geq 0$, we denote the closed r-neighborhood of a Borel set Ω in the complex plane $\mathbb{C}$, i.e. $O_r(\Omega) = \{z \in \mathbb{C} \mid \mathrm{dist}(z, \Omega) \leq r\}$.

2. OPERATOR RICCATI EQUATION

We start by recalling the concepts of weak, strong, and operator solutions to the operator Riccati equation (see [5, 6]).

Definition 2.1. Assume that A_0 and A_1 are possibly unbounded densely defined closed operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Let B and C be bounded operators from $\mathfrak{H}_1$ to $\mathfrak{H}_0$ and from $\mathfrak{H}_0$ to $\mathfrak{H}_1$, respectively.

A bounded operator $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ is said to be a weak solution of the Riccati equation

$$KA_0 - A_1 K + KBK = C \tag{2.1}$$

if

$$(KA_0 x, y) - (Kx, A_1^* y) + (KBKx, y) = (Cx, y)$$

$$\text{for all } x \in \text{Dom}(A_0) \text{ and } y \in \text{Dom}(A_1^*).$$

A bounded operator $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ is called a strong solution of the Riccati equation (2.1) if

$$\text{Ran}\big(K|_{\text{Dom}(A_0)}\big) \subset \text{Dom}(A_1) \tag{2.2}$$

and

$$KA_0 x - A_1 Kx + KBKx = Cx \quad \text{for all} \quad x \in \text{Dom}(A_0). \tag{2.3}$$

Finally, $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ is said to be an operator solution of the Riccati equation (2.1) if

$$\text{Ran}(K) \subset \text{Dom}(A_1), \tag{2.4}$$

the operator KA_0 is bounded on $\text{Dom}(KA_0) = \text{Dom}(A_0)$, and the equality

$$\overline{KA_0} - A_1 K + KBK = C \tag{2.5}$$

holds as an operator equality, where $\overline{KA_0}$ denotes the closure of KA_0.

Remark 2.2. We will call the equation

$$XA_1^* - A_0^* X - XB^* X = -C^* \tag{2.6}$$

the adjoint of the operator Riccati equation (2.1). It immediately follows from the definition that an operator $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ is a weak solution to the Riccati equation (2.1) if and only if the adjoint of K, $X = K^*$, is a weak solution to the adjoint equation (2.6).

Clearly, any operator solution $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ to the Riccati equation (2.1) is automatically a strong solution. Similarly, any strong solution is also a weak solution. But, in fact, by a result of [6] one does not need to distinguish between weak and strong solutions to the Riccati equation (2.1). This is seen from the following statement.

Lemma 2.3 ([6], Lemma 5.2)**.** *Let A_0 and A_1 be densely defined possibly unbounded closed operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively, and $B \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$, $C \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$. If $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ is a weak solution of the Riccati equation* (2.1) *then K is also a strong solution of* (2.1)*.*

If the operators A_0, A_1, B, and C are as in Definition 2.1 then a 2×2 operator block matrix

$$L = \begin{pmatrix} A_0 & B \\ C & A_1 \end{pmatrix}, \qquad \text{Dom}(L) = \text{Dom}(A_0) \oplus \text{Dom}(A_1), \tag{2.7}$$

is a densely defined and possibly unbounded closed operator on the Hilbert space

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1. \tag{2.8}$$

The operator L will often be viewed as the result of the perturbation of the block diagonal matrix

$$A = \operatorname{diag}(A_0, A_1), \qquad \operatorname{Dom}(A) = \operatorname{Dom}(A_0) \oplus \operatorname{Dom}(A_1), \tag{2.9}$$

by the off-diagonal bounded perturbation

$$V = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{2.10}$$

The operator Riccati equation (2.1) and the block operator matrix L are usually said to be associated to each other. Surely, one can also associate with the matrix L another operator Riccati equation,

$$K'A_1 - A_0K' + K'CK' = B, \tag{2.11}$$

assuming that a solution K' (if it exists) should be a bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}_0$.

It is well known that the solutions to the Riccati equations (2.1) and (2.11) determine invariant subspaces for the operator matrix L (see, e.g., [5] for the case where the matrix L is self-adjoint or [31] for the case of a non-self-adjoint L). These subspaces have the form of the graphs

$$\mathscr{G}(K) = \{x \in \mathfrak{H}_0 \oplus \mathfrak{H}_1 \,|\, x = x_0 \oplus Kx_0 \text{ for some } x_0 \in \mathfrak{H}_0\} \tag{2.12}$$

and

$$\mathscr{G}(K') = \{x \in \mathfrak{H}_0 \oplus \mathfrak{H}_1 \,|\, x = K'x_1 \oplus x_1 \text{ for some } x_1 \in \mathfrak{H}_1\} \tag{2.13}$$

of the corresponding (bounded) solutions K and K'. Notice that the subspaces of the form (2.12) and (2.13) are usually called the graph subspaces associated with the operators K and K', respectively, while K and K' themselves are called the angular operators. Usage of the latter term is explained, in particular, by the fact that if a subspace $\mathfrak{G} \subset \mathfrak{H}$ is a graph $\mathfrak{G} = \mathscr{G}(K)$ of a bounded linear operator K from a subspace $\mathfrak{M}$ to its orthogonal complement $\mathfrak{M}^\perp$, $\mathfrak{M}^\perp = \mathfrak{H} \ominus \mathfrak{M}$, then the following equality holds (see [25]; cf. [17] and [21]):

$$|K| = \tan \Theta(\mathfrak{M}, \mathfrak{G}), \tag{2.14}$$

where $|K|$ is the absolute value of K, $|K| = \sqrt{K^*K}$, and $\Theta(\mathfrak{M}, \mathfrak{G})$ the operator angle (1.5) between the subspaces $\mathfrak{M}$ and $\mathfrak{G}$.

The precise statement relating solutions of the Riccati equations (2.1) and (2.11) to invariant subspaces of the operator matrix (2.7) is as follows.

Lemma 2.4. *Let the entries A_0, A_1, B, and C be as in Definition 2.1 and let a 2×2 block operator matrix L be given by* (2.7). *Then the graph $\mathscr{G}(K)$ of a bounded operator K from $\mathfrak{H}_0$ to $\mathfrak{H}_1$ satisfying* (2.2) *is an invariant subspace for the operator matrix L if and only if K is a strong solution to the operator Riccati equation* (2.1). *Similarly, the graph $\mathscr{G}(K')$ of an operator $K' \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ such that $\operatorname{Ran}(K'|_{\operatorname{Dom}(A_1)}) \subset \operatorname{Dom}(A_0)$ is an invariant subspace for L if and only if this operator is a strong solution to the Riccati equation* (2.11).

The proof of this lemma is straightforward and follows the same line as the proof of the corresponding part in [5, Lemma 5.3]. Thus, we omit it.

The next assertion contains two useful identities involving the strong solutions to the Riccati equations (2.1) and (2.11).

Lemma 2.5. *Let the entries A_0, A_1, B, and C be as in Definition 2.1. Assume that operators $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ and $K' \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ are strong solutions to equations* (2.1) *and* (2.11), *respectively. Then*

$$\operatorname{Ran}(K'K|_{\operatorname{Dom}(A_0)}) \subset \operatorname{Dom}(A_0), \quad \operatorname{Ran}(KK'|_{\operatorname{Dom}(A_1)}) \subset \operatorname{Dom}(A_1), \tag{2.15}$$

and

$$(I - K'K)(A_0 + BK)x = (A_0 - K'C)(I - K'K)x \quad \text{for all } x \in \text{Dom}(A_0), \qquad (2.16)$$

$$(I - KK')(A_1 + CK')y = (A_1 - KB)(I - KK')y \quad \text{for all } y \in \text{Dom}(A_1). \qquad (2.17)$$

Proof. The inclusions (2.15) follow immediately from the definition of a strong solution to the operator Riccati equation (see condition (2.2)).

Let $x \in \text{Dom}(A_0)$. Taking into account the first of the inclusions (2.15) as well as the inclusions $\text{Ran}(K|_{\text{Dom}(A_0)}) \subset \text{Dom}(A_1)$ and $\text{Ran}(K'|_{\text{Dom}(A_1)}) \subset \text{Dom}(A_0)$ one can write

$$(A_0 - K'C)(I - K'K)x = (A_0 - K'C)x - (A_0K' - K'CK')Kx$$
$$= (A_0 - K'C)x - (K'A_1 - B)Kx, \qquad (2.18)$$

by making use of the Riccati equation (2.11) itself at the second step. Similarly,

$$(I - K'K)(A_0 + BK)x = (A_0 + BK)x - K'(KA_0 + KBK)x$$
$$= (A_0 + BK)x - K'(C + A_1K)x, \qquad (2.19)$$

due to the Riccati equation (2.1). Comparing (2.18) and (2.19) we arrive at the identity (2.16).

Identity (2.17) is proven analogously. $\qquad \square$

We will also need the following auxiliary lemma.

Lemma 2.6. *Suppose that the operators $K \in \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ and $K' \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ are such that the 2×2 operator block matrix*

$$W = \begin{pmatrix} I & K' \\ K & I \end{pmatrix} \qquad (2.20)$$

considered on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ is boundedly invertible, i.e. the inverse operator W^{-1} exists and is bounded. Then the graphs $\mathscr{G}(K)$ and $\mathscr{G}(K')$ of the operators K and K' are linearly independent subspaces of $\mathfrak{H}$ and

$$\mathfrak{H} = \mathscr{G}(K) \dotplus \mathscr{G}(K'), \qquad (2.21)$$

where the sign "$\dotplus$" denotes the direct sum of two subspaces.

Proof. The existence and boundedness of W^{-1} imply that equation $Wx = y$ is uniquely solvable for any $y \in \mathfrak{H}$. This means that there are unique $x_0 \in \mathfrak{H}_0$ and unique $x_1 \in \mathfrak{H}_1$ such that $y = x_0 \oplus Kx_0 + K'x_1 \oplus x_1$ and hence $\mathfrak{H} \subset \mathscr{G}(K) + \mathscr{G}(K')$. Since both $\mathscr{G}(K)$ and $\mathscr{G}(K')$ are subspaces of $\mathfrak{H}$, the inclusion turns into equality, $\mathfrak{H} = \mathscr{G}(K) + \mathscr{G}(K')$. The linear independence of $\mathscr{G}(K)$ and $\mathscr{G}(K')$ follows from the fact that equation $Wx = 0$ has only the trivial solution $x = 0$. $\qquad \square$

Remark 2.7. It is well known that the following three statements are equivalent.

 (i) The operator matrix (2.20) is boundedly invertible.
 (ii) The inverse $(I - KK')^{-1}$ exists and is bounded.
 (iii) The inverse $(I - K'K)^{-1}$ exists and is bounded

For a proof of this assertion see, e.g. [22, Theorem 1.1. and Lemma 2.1] where even a Banach-space case of 2×2 block operator matrices of the form (2.20) with unbounded entries K and K' has been studied.

Remark 2.8. The inverse of the operator W is explicitly written as

$$W^{-1} = \begin{pmatrix} (I - K'K)^{-1} & -K'(I - KK')^{-1} \\ -K(I - K'K)^{-1} & (I - KK')^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (I-K'K)^{-1} & -(I-K'K)^{-1}K' \\ -(I-KK')^{-1}K & (I-KK')^{-1}. \end{pmatrix} \tag{2.22}$$

The (oblique) projections $Q_{\mathscr{G}(K)}$ and $Q_{\mathscr{G}(K')}$ onto the graph subspaces $\mathscr{G}(K)$ and $\mathscr{G}(K')$ along the corresponding complementary graph subspaces $\mathscr{G}(K')$ and $\mathscr{G}(K)$ are given by

$$Q_{\mathscr{G}(K)} = \begin{pmatrix} I \\ K \end{pmatrix}(I-K'K)^{-1}\begin{pmatrix} I & -K' \end{pmatrix} \text{ and } Q_{\mathscr{G}(K')} = \begin{pmatrix} K' \\ I \end{pmatrix}(I-KK')^{-1}\begin{pmatrix} -K & I \end{pmatrix}, \tag{2.23}$$

respectively.

Corollary 2.9. *Assume the hypothesis of Lemma 2.4. Suppose that $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ and $K' \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ are strong solutions to the Riccati equations (2.1) and (2.11), respectively. Assume, in addition, that the 2×2 operator block matrix W formed with these solutions according to (2.20) is a boundedly invertible operator on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. Then:*

(i) *The operator L is similar to a block diagonal operator matrix $Z = \mathrm{diag}(Z_0, Z_1)$,*

$$L = WZW^{-1}, \tag{2.24}$$

where Z_0 and Z_1 are operators on $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively, given by

$$Z_0 = A_0 + BK, \quad \mathrm{Dom}(Z_0) = \mathrm{Dom}(A_0), \tag{2.25}$$

$$Z_1 = A_1 + CK', \quad \mathrm{Dom}(Z_1) = \mathrm{Dom}(A_1). \tag{2.26}$$

(ii) *The Hilbert space $\mathfrak{H}$ splits into the direct sum $\mathfrak{H} = \mathfrak{H}_0' \dotplus \mathfrak{H}_1'$ of the graph subspaces $\mathfrak{H}_0' = \mathscr{G}(K)$ and $\mathfrak{H}_1' = \mathscr{G}(K')$ that are invariant under L. The restrictions $L|_{\mathfrak{H}_0'}$ and $L|_{\mathfrak{H}_1'}$ of L onto $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are similar to the operators Z_0 and Z_1,*

$$W_0^{-1}L|_{\mathfrak{H}_0'}W_0 = Z_0 \quad and \quad W_1^{-1}L|_{\mathfrak{H}_1'}W_1 = Z_1, \tag{2.27}$$

where the entries $W_0 : \mathfrak{H}_0 \to \mathfrak{H}_0'$ and $W_1 : \mathfrak{H}_1 \to \mathfrak{H}_1'$ correspond to the respective columns of the block operator matrix W,

$$W_0 x_0 = \begin{pmatrix} I \\ K \end{pmatrix}x_0, \ x_0 \in \mathfrak{H}_0, \quad and \quad W_1 x_1 = \begin{pmatrix} K' \\ I \end{pmatrix}x_1, \ x_1 \in \mathfrak{H}_1. \tag{2.28}$$

Proof. First, one verifies by inspection that $LW = WZ$ taking to account that K and K' are the strong solutions to the Riccati equations (2.1) and (2.11), respectively. The remaining statements immediately follow from Lemma 2.4 combined with Lemma 2.6. $\qquad\square$

Remark 2.10. The similarity (2.24) of the operators L and Z implies that the spectrum of L coincides with the union of the spectra of Z_0 and Z_1, that is, $\mathrm{spec}(L) = \mathrm{spec}(Z_0) \cup \mathrm{spec}(Z_1)$.

3. OPERATOR SYLVESTER EQUATION

Along with the Riccati equation (2.1) we need to consider the operator Sylvester equation

$$XA_0 - A_1X = Y \tag{3.1}$$

assuming that the entries A_0 and A_1 are as in Definition 2.1 and $Y \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$. The Sylvester equation is a particular (linear) case of the Riccati equation and its weak, strong, and operator solutions $X \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ are understood in the same way as in the above definition. Furthermore, by Lemma 2.3 (cf. [8, Lemma 1.3]) one does not need to distinguish between the weak and strong solutions to (3.1).

Because of its importance for various areas of mathematics there is an enormous literature on the Sylvester equation (for a review and many references see paper [14]). With equation (3.1)

one often associates the Sylvester operator S defined on the Banach space $\mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ by the left-hand side of (3.1):

$$S(X) = XA_0 - A_1X \tag{3.2}$$

with domain

$$\mathrm{Dom}(S) = \left\{ X \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1) \mid \mathrm{Ran}(X\big|_{\mathrm{Dom}(A_0)}) \subset \mathrm{Dom}(A_1) \right\}. \tag{3.3}$$

Clearly, the Sylvester equation (3.1) has a unique solution $X \in \mathrm{Dom}(S)$ if and only if $0 \notin \mathrm{spec}(S)$. It is known that in general the spectrum of S is larger than the (numerical) difference between the spectra of A_0 and A_1. More precisely, provided that $\mathrm{spec}(A_0) \neq \mathbb{C}$ or $\mathrm{spec}(A_1) \neq \mathbb{C}$ always the following inclusion holds [8]:

$$\overline{\mathrm{spec}(A_0) - \mathrm{spec}(A_1)} \subset \mathrm{spec}(S), \tag{3.4}$$

where we use the notation $\Sigma - \Delta = \{z - \zeta \mid z \in \Sigma, \zeta \in \Delta\}$ for the numerical difference between two Borel subsets Σ and Δ of the complex plane $\mathbb{C}$. The opposite inclusion in (3.4) may fail to hold if both operators A_0 and A_1 are unbounded. The corresponding example was first given by V. Q. Phóng [43] for the Sylvester equation (3.1) where one of the entries A_0 and A_1 is an operator on a Banach (but not Hilbert) space. An example where both A_0 and A_1 are operators on Hilbert spaces and $\mathrm{spec}(S) \not\subset \overline{\mathrm{spec}(A_0) - \mathrm{spec}(A_1)}$ may be found in [8, Example 6.2]. Equality

$$\mathrm{spec}(S) = \mathrm{spec}(A_0) - \mathrm{spec}(A_1) \tag{3.5}$$

holds if both A_0 and A_1 are bounded operators. This result is due to G. Lumer and M. Rosenblum [34]. Equality (3.5) also holds if only one of the entries A_0 and A_1 is a bounded operator [8]. In this case (3.5) implies that if the spectra A_0 and A_1 are disjoint then $0 \notin \mathrm{spec}(S)$ and hence the operator S is boundedly invertible. Moreover, a unique solution of the Sylvester equation (3.1) admits an "explicit" representation in the form a contour integral.

Lemma 3.1. *Let A_0 be a possibly unbounded densely defined closed operator on the Hilbert space $\mathfrak{H}_0$ and A_1 a bounded operator on the Hilbert space $\mathfrak{H}_1$ such that*

$$\mathrm{spec}(A_0) \cap \mathrm{spec}(A_1) = \varnothing$$

and $Y \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$. Then the Sylvester equation (3.1) has a unique operator solution

$$X = \frac{1}{2\pi\mathrm{i}} \int_\gamma dz\, (A_1 - z)^{-1} Y (A_0 - z)^{-1}, \tag{3.6}$$

where γ is a union of closed contours in $\mathbb{C}$ with total winding numbers 0 around $\mathrm{spec}(A_0)$ and 1 around $\mathrm{spec}(A_1)$ and the integral converges in the norm operator topology.

Corollary 3.2. *Under the hypothesis of Lemma 3.1 the norm of the inverse of the Sylvester operator S may be estimated as*

$$\|S^{-1}\| \leq (2\pi)^{-1} |\gamma| \sup_{z \in \gamma} \|(A_0 - z)^{-1}\| \, \|(A_1 - z)^{-1}\|,$$

where $|\gamma|$ denotes the length of the contour γ in (3.6).

The result of Lemma 3.1 may be attributed to M. G. Krein who lectured on the operator Sylvester equation in late 1940s (see [14]). Later, it was independently obtained by M. Rosenblum [45].

As for the Sylvester operator (3.2) with both unbounded entries A_0 and A_1, we have an important result which is due to W. Arendt, F. Räbiger, and A. Sourour (see [8, Theorem 4.1 and Corollary 5.4]).

Theorem 3.3 ([8])**.** *Let A_0 and A_1 be closed densely defined operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Assume that one (or both) of the following holds (hold) true:*

   (i) *A_0 and $(-A_1)$ are generators of eventually norm continuous C_0-semigroups;*
   (ii) *A_0 and $(-A_1)$ are generators of C_0-semigroups one of which is holomorphic.*

Then the spectrum of the Sylvester operator (3.2) is given by (3.5).

Recall that an operator H on the Hilbert space $\mathfrak{M}$ is said to be *m-dissipative* if it is closed and both the spectrum and numerical range of H are contained in the left half-plane $\{z \in \mathbb{C} \,|\, \operatorname{Im} z \leq 0\}$. The Lumer-Phillips theorem asserts (see, e.g., [19, Section II.3.b]; cf. [20, Theorem B.21]) that a C_0-semigroup on $\mathfrak{M}$ is a contraction semigroup if and only if its generator is an *m-dissipative* operator. The next statement represents a generalization of a well known result by E. Heinz ([23, Satz 5]) to the case of unbounded operators. Notice that the exponential e^{Ht}, $t \geq 0$, is understood below as the corresponding element of the strongly continuous contraction semigroup generated by an (unbounded) *m-dissipative* operator H.

Theorem 3.4. *Let $A_0 + \frac{\delta}{2}I$ and $-A_1 + \frac{\delta}{2}I$, $\delta > 0$, be m-dissipative operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively, and $Y \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$. Then the Sylvester equation (3.1) has a unique weak (and hence unique strong) solution given by*

$$X = -\int_0^{+\infty} dt \, \mathrm{e}^{-A_1 t} Y \mathrm{e}^{A_0 t}, \tag{3.7}$$

where the integral is understood in the weak operator topology. Moreover, the norm of the solution (3.7) satisfies the estimate

$$\|X\| \leq \frac{1}{\delta} \|Y\|. \tag{3.8}$$

Proof. Under the hypothesis the operators A_0 and $(-A_1)$ are themselves *m-dissipative*. Let $U_0(t)$ and $U_1(t)$, $t \geq 0$, be contraction C_0-semigroups generated respectively by A_0 and $(-A_1)$, that is, $U_0(t) = \mathrm{e}^{A_0 t}$ and $U_1(t) = \mathrm{e}^{-A_1 t}$. Clearly,

$$\|U_0(t)\| \leq \mathrm{e}^{-\frac{\delta}{2}t} \text{ and } \|U_1(t)\| \leq \mathrm{e}^{-\frac{\delta}{2}t}, \quad t \geq 0. \tag{3.9}$$

The same bound also holds for the adjoint semigroup $U_1(t)^*$, $t \geq 0$, whose generator is the *m-dissipative* operator $(-A_1^*)$. Pick up arbitrary $x \in \mathfrak{H}_0$ and $y \in \mathfrak{H}_1$ and introduce the orbit maps $t \mapsto \xi_x(t) = U_0(t)x$ and $\zeta_y : t \mapsto \zeta_y(t) = U_1(t)^*y$. By the definition of a strongly continuous semigroup, these maps are continuous functions of $t \in [0, \infty)$. Taking into account the bounds (3.9) one then concludes that the improper integral

$$\int_0^\infty dt \, (U_1(t) Y U_0(t)x, y) = \int_0^\infty dt \, (Y\xi_x(t), \zeta_y(t)) \tag{3.10}$$

converges and its absolute value is bounded by $\|Y\|\|x\|\|y\|/\delta$. Thus, the weak integral on the right-hand side of (3.7) exists and the bound (3.8) holds.

Now assume that $x \in \operatorname{Dom}(A_0)$ and $y \in \operatorname{Dom}(A_1^*)$. In this case the orbit maps $\xi_x(t)$ and $\zeta_y(t)$ are continuously differentiable in t and $\frac{d}{dt}\xi_x(t) = U_0(t)A_0 x$, $\frac{d}{dt}\zeta_y(t) = -U_1(t)^*A_1^* y$. For X given by (3.7), an elementary computation shows that

$$(XA_0 x, y) - (Xx, A_1^* y) = -\int_0^\infty dt \left((Y\tfrac{d}{dt}\xi_x(t), \zeta_y(t)) + (Y\xi_x(t), \tfrac{d}{dt}\zeta_y(t)) \right)$$

$$= -\int_0^\infty dt \, \tfrac{d}{dt}(Y\xi_x(t), \zeta_y(t)) = (Y\xi(0), \zeta(0)),$$

taking into account (3.9) in the last step. Since $\xi_x(0) = x$ and $\zeta_y(0) = y$, by Definition 2.1 this implies that the integral (3.7) is a weak (and hence strong) solution to the Sylvester equation (3.1).

To prove the uniqueness of the weak solution (3.7) it is sufficient to show that the homogeneous Sylvester equation $XA_0 - A_1X = 0$ has the only weak solution $X = 0$. For a weak solution X to this equation we have

$$(XA_0u, v) - (Xu, A_1^*v) = 0 \quad \text{for all } u \in \text{Dom}(A_0) \text{ and } v \in \text{Dom}(A_1^*). \tag{3.11}$$

Take the vectors u and v of the form $u = \xi_x(t) = e^{A_0 t}x$, $v = \zeta_y(t) = e^{-A_1^* t}y$, $t \geq 0$, where the orbit maps $\xi_x(t)$ and $\zeta_y(t)$ correspond to some $x \in \text{Dom}(A_0)$ and $y \in \text{Dom}(A_1^*)$ and hence are both continuously differentiable in $t \in [0, \infty)$. Notice that the assumption $x \in \text{Dom}(A_0)$, $y \in \text{Dom}(A_1^*)$ also implies (see, e.g. [19, Lemma 1.3]) that $u \in \text{Dom}(A_0)$, $v \in \text{Dom}(A_1)$, and $A_0u = \frac{d}{dt}\xi_x(t)$, $A_1^*v = -\frac{d}{dt}\zeta(t)$. With such a choice of u and v it follows from (3.11) that

$$\frac{d}{dt}(X\xi_x(t), \zeta_y(t)) = 0 \quad \text{whenever } x \in \text{Dom}(A_0) \text{ and } y \in \text{Dom}(A_1^*).$$

Hence the function $(X\xi_x(t), \zeta_y(t))$, $t \geq 0$, is a constant. Moreover, it equals zero since it vanishes as $t \to \infty$. This yields in particular that

$$(Xx, y) = (X\xi_x(0), \zeta_y(0)) = 0 \quad \text{for all } x \in \text{Dom}(A_0) \text{ and } y \in \text{Dom}(A_1^*).$$

The latter implies $X = 0$, which completes the proof. $\qquad\square$

Remark 3.5. A statement similar to Theorem 3.4 was previously announced without a proof in [5] (see [5, Lemma 2.6]).

The second important example where a bound of the (3.8) type exists is given in [13, Theorem 3.2]. This example is as follows.

Theorem 3.6. *Assume that the operators A_0 and A_1 are densely defined and closed. Assume, in addition, that there is $\lambda \in \rho(A_1)$ such that $\|A_0 - \lambda\| \leq r$ and $\|(A_1 - \lambda)^{-1}\| \leq (r + \delta)^{-1}$ for some $r \geq 0$ and $\delta > 0$. Then for any $Y \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ the unique strong solution X to the Sylvester equation (3.1) admits the estimate $\delta\|X\| \leq \|Y\|$.*

If the operators A_0 and A_1 are normal then no reference point λ is needed and the result is stated in a more universal form (see [13, Theorem 3.2]).

Corollary 3.7. *Let both A_0 and A_1 be normal operators such that $\text{spec}(A_0)$ is contained in a closed disk of radius r, $r \geq 0$, while $\text{spec}(A_1)$ is disjoint from the open disk (with the same center) of radius $r + \delta$, $\delta > 0$. Then for any $Y \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ the Sylvester equation (3.1) has a unique strong solution X and $\delta\|X\| \leq \|Y\|$.*

The above two theorems and the corollary give examples where the bounded inverse of the Sylvester operator S exists and for the norm of S^{-1} the estimate $\delta\|S^{-1}\| \leq 1$ holds with some $\delta > 0$. Moreover, this estimate is universal in the sense that it remains valid for any A_0 and A_1 satisfying the corresponding hypotheses.

4. EXISTENCE RESULTS FOR THE RICCATI EQUATION

In this section we return to the operator Riccati equation (2.1) to prove some sufficient conditions for its solvability. In their proof we will rely just on the assumption that an estimate like (3.8) holds for the solution of the corresponding Sylvester equation.

Theorem 4.1. *Let A_0 and A_1 be possibly unbounded closed densely defined operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively. Assume that the Sylvester operator S defined on $\mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ by (3.2) and (3.3) is boundedly invertible (that is, $0 \notin \operatorname{spec}(S)$) and*

$$\|S^{-1}\| \leq \frac{1}{\delta} \tag{4.1}$$

for some $\delta > 0$. Assume, in addition, the operators $B \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ and $C \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ are such that the following bound holds:

$$\sqrt{\|B\|\|C\|} < \frac{\delta}{2}. \tag{4.2}$$

Then the Riccati equation (2.1) has a unique strong solution in the ball $\mathscr{O}_{\delta/(2\|B\|)}(\mathfrak{H}_1, \mathfrak{H}_0)$. The strong solution K satisfies the estimate

$$\|K\| \leq \frac{\|C\|}{\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \|B\|\,\|C\|}}. \tag{4.3}$$

Proof. If $B = 0$ then the assertion, including the estimate (4.3), follows immediately from the hypothesis on the invertibility of S on $\mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ taking into account the bound (4.1).

Suppose that $B \neq 0$. In this case the proof is performed by applying Banach's Fixed Point Theorem. First, we notice that the bounded invertibility of S on $\mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ allows us to rewrite the Riccati equation (2.1) in the form

$$K = F(K)$$

where the mapping $F : \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1) \to \operatorname{Dom}(S)$ is given by

$$F(K) = S^{-1}(C - KBK).$$

By (4.1) we have

$$\|F(K)\| \leq \frac{1}{\delta}(\|C\| + \|B\|\,\|K\|^2), \quad K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1) \tag{4.4}$$

and

$$\|F(K_1) - F(K_2)\| \leq \frac{1}{\delta}\|B\|\,(\|K_1\| + \|K_2\|)\,\|K_1 - K_2\|, \quad K_1, K_2 \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1). \tag{4.5}$$

The bound (4.4) implies that F maps the ball $\mathscr{O}_r(\mathfrak{H}_0, \mathfrak{H}_1)$ into itself whenever

$$\|B\|\,r^2 + \|C\| \leq r\delta. \tag{4.6}$$

At the same time, from (4.5) it follows that F is a strict contraction of the ball $\mathscr{O}_r(\mathfrak{H}_1, \mathfrak{H}_0)$ whenever

$$2\|B\|r < \delta. \tag{4.7}$$

Solving inequalities (4.6) and (4.7) one concludes that if the radius r of the ball $\mathscr{O}_r(\mathfrak{H}_1, \mathfrak{H}_0)$ is within the bounds

$$\frac{\|C\|}{\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \|B\|\,\|C\|}} \leq r < \frac{\delta}{2\|B\|}, \tag{4.8}$$

then F is a strictly contractive mapping of the ball $\mathscr{O}_r(\mathfrak{H}_1, \mathfrak{H}_0)$ into itself. Applying Banach's Fixed Point Theorem one then infers that equation (2.1) has a unique solution within any ball $\mathscr{O}_r(\mathfrak{H}_1, \mathfrak{H}_0)$ whenever the radius r satisfies (4.8). This means that the fixed point is the same for all the radii satisfying (4.8) and hence it belongs to the smallest of the balls. This conclusion proves the bound (4.3) and completes the whole proof. $\qquad\square$

Remark 4.2. In (4.2)–(4.3) one may set $\delta = \|S^{-1}\|^{-1}$.

Remark 4.3. By using the hyperbolic tangent function and its inverse the bound (4.3) (for $B \neq 0$) can be equivalently written in the hypertrigonometric form

$$\|K\| \leq \sqrt{\frac{\|C\|}{\|B\|}} \, \tanh\left(\frac{1}{2}\operatorname{arctanh} \frac{2\sqrt{\|B\|\|C\|}}{\delta}\right). \tag{4.9}$$

Notice that under condition (4.2) we always have

$$\tanh\left(\frac{1}{2}\operatorname{arctanh} \frac{2\sqrt{\|B\|\|C\|}}{\delta}\right) < 1.$$

Remark 4.4. Fixed-point based approaches to prove the solvability of the operator Riccati equation with bounded entries A_0 and A_1 have been used in many papers (see, e.g., [2], [18], [42], [46], [47]). In the case where at least one of the entries A_0 and A_1 is an unbounded self-adjoint or normal operator, a fixed-point approach has been employed in [5], [6], [36], and [40]. Theorem 4.1 represents an extension of the fixed-point existence results obtained in [47, Theorem 3.5] and [42, Theorem 3.1] for the Riccati equation (2.1) with both bounded A_0 and A_1 to the case where the entries A_0 and A_1 are not necessarily bounded.

Theorem 4.5. *Assume the hypothesis of Theorem 4.1. Then the block operator matrix L defined by (2.7) is block diagonalizable with respect to the direct sum decomposition $\mathfrak{H} = \mathscr{G}(K) \dotplus \mathscr{G}(K')$ where K is the unique strong solution to the Riccati equation (2.1) within the operator ball $\mathscr{O}_{\delta/(2\|B\|)}(\mathfrak{H}_0, \mathfrak{H}_1)$ and K' the unique strong solution to the Riccati equation (2.11) within the operator ball $\mathscr{O}_{\delta/(2\|C\|)}(\mathfrak{H}_1, \mathfrak{H}_0)$.*

Proof. By Theorem 4.1 for K the estimate (4.3) holds. By the same theorem for K' we have

$$\|K'\| \leq \|B\| \left(\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \|B\|\,\|C\|}\right)^{-1}. \tag{4.10}$$

Then the hypothesis $\|B\|\|C\| < \delta/2$ also implies that $\|K\|\|K'\| < 1$. Hence by Remark 2.7 the operator W in (2.20) is boundedly invertible. Applying Corollary 2.9 completes the proof. $\square$

Remark 4.6. If, in addition, both operators A_0 and A_1 are normal then, for r defined by

$$r = \frac{\|B\|\,\|C\|}{\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \|B\|\,\|C\|}} = \sqrt{\|B\|\,\|C\|}\, \tanh\left(\frac{1}{2}\operatorname{arctanh} \frac{2\sqrt{\|B\|\|C\|}}{\delta}\right), \tag{4.11}$$

the spectrum of the block matrix L lies in the closed r-neighborhood of the spectrum of its main-diagonal part $A = \operatorname{diag}(A_0, A_1)$. That is, $\operatorname{dist}(z, \operatorname{spec}(A_0) \cup \operatorname{spec}(A_1)) \leq r$ whenever $z \in \operatorname{spec}(L)$. This immediately follows from the representation (2.24)–(2.26) and the bounds (4.3) and (4.10) (see also Remark 2.10). Notice that if $B \neq 0$ and $C \neq 0$ then $r < \sqrt{\|B\|\|C\|}$ and hence $r < \|V\|$ taking into account that $\|V\| = \max(\|B\|, \|C\|)$.

From now on we assume that the entries A_0 and A_1 are self-adjoint operators with disjoint spectra and thus adopt the following

Hypothesis 4.7. *Let A_0 and A_1 be (possibly unbounded) self-adjoint operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ with domains $\operatorname{Dom}(A_1)$ and $\operatorname{Dom}(A_1)$, respectively. Assume that the spectra of the operators A_0 and A_1 are disjoint and let*

$$d = \operatorname{dist}(\operatorname{spec}(A_0), \operatorname{spec}(A_1)) \, (> 0). \tag{4.12}$$

Hypothesis 4.7 imposes no restrictions on the mutual position of the spectral sets $\text{spec}(A_0)$ and $\text{spec}(A_1)$ except that they are disjoint and separated from each other by a distance d. Sometimes, however, we will consider particular spectral dispositions described in

Hypothesis 4.8. *Assume Hypothesis 4.7. Assume, in addition, that either the spectra of A_0 and A_1 are subordinated, that is,*

$$\sup \text{spec}(A_0) < \inf \text{spec}(A_1) \ \text{ or } \ \inf \text{spec}(A_0) > \sup \text{spec}(A_1), \tag{4.13}$$

or one of the sets $\text{spec}(A_0)$ and $\text{spec}(A_1)$ lies in a finite gap of the other set, that is,

$$\text{conv}\big(\text{spec}(A_0)\big) \cap \text{spec}(A_1) = \varnothing \ \text{ or } \ \text{spec}(A_0) \cap \text{conv}\big(\text{spec}(A_1)\big) = \varnothing. \tag{4.14}$$

Under Hypotheses 4.7 or 4.8 the bound on the norm of the inverse of the Sylvester operator (3.2) may be given in terms of the distance d between $\text{spec}(A_0)$ and $\text{spec}(A_1)$. The following result is well known.

Theorem 4.9. *Assume Hypothesis 4.7. Let the Sylvester operator S be defined by (3.2) and (3.3).*

(i) *Then the inverse of S exists and is bounded. Moreover, the following estimate holds:*

$$\|S^{-1}\| \le \frac{\pi}{2d}. \tag{4.15}$$

(ii) *Assume Hypothesis 4.8. Then the following stronger inequality holds:*

$$\|S^{-1}\| \le \frac{1}{d}. \tag{4.16}$$

Remark 4.10. In the generic case (i), where no assumptions on the mutual position of the sets $\text{spec}(A_0)$ and $\text{spec}(A_1)$ are imposed, the existence of a universal constant c such that $\|S^{-1}\| \le \dfrac{c}{d}$ has been proven in [13]. The proof of the fact that $c = \pi/2$ is best possible is due to R. McEachin [35]. For more details see [5, Remark 2.8]. As for the particular spectral disposition (4.13), the bound (4.16) is an immediate corollary to Theorem 3.4. Since any self-adjoint operator is simultaneously a normal operator, in the case of the spectral disposition (4.14) the bound (4.16) follows from Corollary 3.7. Sharpness of the bound (4.16) in case (ii) is proven by an elementary example where the spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are one-dimensional, $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}$, and the entries $A_0 = a_0$ and $A_1 = a_1$ are real numbers such that $|a_1 - a_0| = d > 0$.

Under the assumption that both the entries A_0 and A_1 are self-adjoint operators, below we present an existence result for the operator Riccati equation (2.1), which is written directly in terms of the distance between the spectra of the entries A_0 and A_1 (and norms of the operators B and C). The result is an immediate corollary to Theorems 4.1 and 4.9. We only notice that the role of the quantity δ in the bounds like (4.1) (see inequalities (4.18) and (4.20) below) will be played by either $\frac{2}{\pi}d$ from (4.15) or d from (4.16).

Theorem 4.11. *Assume Hypothesis 4.7.*

(i) *Then for any $B \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ and $C \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ such that*

$$\sqrt{\|B\| \|C\|} < \frac{d}{\pi} \tag{4.17}$$

the Riccati equation (2.1) has a unique strong solution K in the ball $\mathscr{O}_{d/(\pi \|B\|)}(\mathfrak{H}_0, \mathfrak{H}_1)$. This solution satisfies the estimate

$$\|K\| \le \frac{\|C\|}{\frac{d}{\pi} + \sqrt{\frac{d^2}{\pi^2} - \|B\| \|C\|}}. \tag{4.18}$$

(ii) *If the conditions of Hypothesis 4.8 also hold then the Riccati equation* (2.1) *has a unique strong solution* K *in the ball* $\mathcal{O}_{d/(2\|B\|)}(\mathfrak{H}_0, \mathfrak{H}_1)$ *whenever* $B \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_0)$ *and* $C \in \mathcal{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ *satisfy the bound*

$$\sqrt{\|B\|\|C\|} < \frac{d}{2}. \tag{4.19}$$

The solution K *satisfies the estimate*

$$\|K\| \le \frac{\|C\|}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \|B\|\,\|C\|}}. \tag{4.20}$$

Remark 4.12. The part (i) is a refinement of Theorem 3.6 in [5] that only claimed the existence of a weak (but not strong) solution to the Riccati equation (2.1) within the ball $\mathcal{O}_{d/(\pi\|B\|)}(\mathfrak{H}_1, \mathfrak{H}_0)$. The result of the part (ii) is new.

Remark 4.13. Let r be given by formula (4.11) where $\delta = \frac{2}{\pi}d$ in case (i) and $\delta = d$ in case (ii). By Remarks 2.10 and 4.6 one concludes that the spectrum of the block operator matrix L consists of a two disjoint components $\sigma_0' = \mathrm{spec}(Z_0)$ and $\sigma_1' = \mathrm{spec}(Z_1)$ lying in the closed r-neighborhoods $O_r(\mathrm{spec}(A_0))$ and $O_r(\mathrm{spec}(A_1))$ of the corresponding spectral sets $\mathrm{spec}(A_0)$ and $\mathrm{spec}(A_1)$.

Remark 4.14. Examples 4.15 and 4.16 below show that the bound (4.20) is sharp in the following sense. Given a number $d > 0$ and values of the norms $\|B\|$ and $\|C\|$ satisfying (4.19) one can always present self-adjoint (and even rank one or two) entries A_0, A_1 and bounded B and C such that in case (ii) the bound (4.20) turns into equality. Notice that Examples 4.15 and 4.16 serve for the spectral dispositions (4.13) and (4.14), respectively.

Example 4.15. Let $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}$. In this case the entries A_0, A_1, B and C of (2.1) are simply the operators of multiplication by numbers. Set $A_0 = -\frac{d}{2}$, $A_1 = \frac{d}{2}$, $B = b$, and $C = -c$ where b, c, and d are positive numbers such that $\sqrt{bc} < d/2$. The Riccati equation (2.1) turns into a numeric quadratic equation whose solutions $K^{(1)}$ and $K^{(2)}$ are given by

$$K^{(1)} = \frac{c}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - bc}}, \quad K^{(2)} = \frac{c}{\frac{d}{2} - \sqrt{\frac{d^2}{4} - bc}}. \tag{4.21}$$

The right-hand sides of the equalities in (4.21) also represent the norms of the corresponding solutions $K^{(1)}$ and $K^{(2)}$. Obviously, only the solution $K^{(1)}$ satisfies the bound $\|K\| < \frac{d}{2\|B\|}$. Also notice that the eigenvalues of the associated 2×2 matrix L (which is given by (2.7)) read $\lambda_- = -\sqrt{d^2/4 - bc}$ and $\lambda_+ = -\lambda_-$. One observes, in particular, that $\lambda_- = A_0 + BK^{(1)}$.

Example 4.16. Let $\mathfrak{H}_0 = \mathbb{C}$ and $\mathfrak{H}_1 = \mathbb{C}^2$. Assume that

$$A_0 = 0, \quad A_1 = \begin{pmatrix} -d & 0 \\ 0 & d \end{pmatrix}, \quad B = (0 \quad b), \quad \text{and} \quad C = \begin{pmatrix} 0 \\ -c \end{pmatrix},$$

where b, c, and d are positive numbers such that $\sqrt{bc} < d/2$. In this case the Riccati equation (2.1) is easily solved explicitly. It has two solutions $K^{(1)} = \begin{pmatrix} k_-^{(1)} \\ k_+^{(1)} \end{pmatrix}$ and $K^{(2)} = \begin{pmatrix} k_-^{(2)} \\ k_+^{(2)} \end{pmatrix}$ with $k_-^{(1)} = k_-^{(2)} = 0$ and

$$k_+^{(1)} = \frac{c}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - bc}}, \quad k_+^{(2)} = \frac{c}{\frac{d}{2} - \sqrt{\frac{d^2}{4} - bc}}.$$

Clearly, $\|B\| = b$, $\|C\| = c$, and only the solution $K^{(1)}$ belongs to the ball $\mathscr{O}_{d/(2\|B\|)}(\mathfrak{H}_1, \mathfrak{H}_0)$. Its norm is given by the equality

$$\|K^{(1)}\| = \frac{\|C\|}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \|B\|\,\|C\|}}.$$

5. J-SYMMETRIC PERTURBATIONS

In this section we deal with perturbations of spectral subspaces of a self-adjoint operator under off-diagonal J-self-adjoint perturbations.

For notational setup we adopt the following hypothesis.

Hypothesis 5.1. *Assume that A_0 and A_1 are self-adjoint operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ with domains $\mathrm{Dom}(A_0)$ and $\mathrm{Dom}(A_1)$, respectively. Let B be a bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}_0$ and $C = -B^*$. Also assume that A and V are operators on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ given by (2.9) and (2.10), respectively, and $L = A + V$ with $\mathrm{Dom}(L) = \mathrm{Dom}(A)$.*

By J,

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{5.1}$$

$\big($cf. (1.3)$\big)$ we denote a natural involution on the Hilbert space $\mathfrak{H}$ associated with its orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. Subsequently introducing the indefinite inner product

$$[x, y] = (Jx, y), \quad x, y \in \mathfrak{H}, \tag{5.2}$$

turns $\mathfrak{H}$ into a Krein space that we denote by $\mathfrak{K}$.

A (closed) subspace $\mathfrak{L} \subset \mathfrak{K}$ is called *uniformly positive* if there is $\gamma > 0$ such that

$$[x, x] \geq \gamma \|x\|^2 \quad \text{for any nonzero } x \in \mathfrak{K}. \tag{5.3}$$

The subspace $\mathfrak{L}$ is called *maximal uniformly positive* if it is not a subset of any other uniformly positive subspace of $\mathfrak{K}$. Uniformly negative and maximal uniformly negative subspaces of $\mathfrak{K}$ are defined in a similar way. The only difference is in the replacement of (5.3) by the inequality $[x, x] \leq -\gamma \|x\|^2$ that should also hold for all $x \in \mathfrak{K}$, $x \neq 0$. For more definitions related to the Krein spaces we refer to [9], [30].

Clearly, under Hypothesis 5.1 both V and L are J-self-adjoint operators on $\mathfrak{H}$, that is, the products JV and JL are self-adjoint with respect to the initial inner product $(\cdot, \cdot)$. This means that V and L are self-adjoint on the Krein space $\mathfrak{K}$.

The statement below provides us with a sufficient condition for a self-adjoint block operator matrix L on $\mathfrak{H}$ to have purely real spectrum and to be similar to a self-adjoint operator on $\mathfrak{H}$. Notice that for the particular case where the spectra of the entries A_0 and A_1 are subordinated, say $\sup \mathrm{spec}(A_0) < \inf \mathrm{spec}(A_1)$, closely related results may be found in [1, Theorem 4.1] and [37, Theorem 3.2].

Theorem 5.2. *Assume Hypothesis 5.1. Suppose that the Riccati equation*

$$KA_0 - A_1K + KBK = -B^* \tag{5.4}$$

has a weak (and hence strong) strictly contractive solution K, $\|K\| < 1$. Then:

(i) *The operator matrix L has a purely real spectrum and it is similar to a self-adjoint operator on $\mathfrak{H}$. In particular, the following equality holds:*

$$L = T\Lambda T^{-1}, \tag{5.5}$$

where T is a bounded and boundedly invertible operator on $\mathfrak{H}$ given by

$$T = \begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \begin{pmatrix} I - K^*K & 0 \\ 0 & I - KK^* \end{pmatrix}^{-1/2} \tag{5.6}$$

and Λ is a block diagonal self-adjoint operator on $\mathfrak{H}$,

$$\Lambda = \mathrm{diag}(\Lambda_0, \Lambda_1), \quad \mathrm{Dom}(\Lambda) = \mathrm{Dom}(\Lambda_0) \oplus \mathrm{Dom}(\Lambda_1), \tag{5.7}$$

whose entries

$$\Lambda_0 = (I - K^*K)^{1/2}(A_0 + BK)(I - K^*K)^{-1/2},$$
$$\mathrm{Dom}(\Lambda_0) = \mathrm{Ran}(I - K^*K)^{1/2}\big|_{\mathrm{Dom}(A_0)}, \tag{5.8}$$

and

$$\Lambda_1 = (I - KK^*)^{1/2}(A_1 - B^*K^*)(I - KK^*)^{-1/2},$$
$$\mathrm{Dom}(\Lambda_1) = \mathrm{Ran}(I - KK^*)^{1/2}\big|_{\mathrm{Dom}(A_1)}, \tag{5.9}$$

are self-adjoint operators on the corresponding component Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$.

(ii) *The graph subspaces $\mathfrak{H}_0' = \mathscr{G}(K)$ and $\mathfrak{H}_1' = \mathscr{G}(K^*)$ are invariant under L and mutually orthogonal with respect to the indefinite inner product (5.2). Moreover, $\mathfrak{K} = \mathfrak{H}_0'[+]\mathfrak{H}_1'$ where the sign "$[+]$" stands for the orthogonal sum in the sense of the Krein space $\mathfrak{K}$. The subspace $\mathfrak{H}_0'$ is maximal uniformly positive while $\mathfrak{H}_1'$ maximal uniformly negative. The restrictions of L onto the subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are $\mathfrak{K}$-unitary equivalent to the self-adjoint operators Λ_0 and Λ_1, respectively.*

Proof. In the case under consideration the second Riccati equation (2.11) associated with the operator matrix L reads

$$K'A_1 - A_0K' - K'B^*K' = B. \tag{5.10}$$

Thus, it simply coincides with the corresponding adjoint (2.6) of the Riccati equation (5.4). By Remark 2.2 this means that the adjoint of K, $K' = K^*$, is a weak (and hence strong) solution to (5.10). Since $\|K^*\| = \|K\| < 1$, the operators $I - K^*K$ and $I - KK^*$ are strictly positive,

$$I - K^*K \geq I - \|K\|^2 > 0 \quad \text{and} \quad I - KK^* \geq I - \|K\|^2 > 0, \tag{5.11}$$

and, hence, boundedly invertible. This also means that the operator T in (5.6) is well defined and bounded. In addition, by Remark 2.7 this implies that the operator W in (2.20) is boundedly invertible and, consequently, the same holds for T.

Now notice that by Lemma 2.5 we have

$$\mathrm{Ran}\big(K^*K|_{\mathrm{Dom}(A_0)}\big) \subset \mathrm{Dom}(A_0), \quad \mathrm{Ran}\big(KK^*|_{\mathrm{Dom}(A_1)}\big) \subset \mathrm{Dom}(A_1),$$

and

$$(I - K^*K)(A_0 + BK)x = (A_0 + K^*B^*)(I - K^*K)x \quad \text{for all } x \in \mathrm{Dom}(A_0), \tag{5.12}$$
$$(I - KK^*)(A_1 - B^*K^*)y = (A_1 - KB)(I - KK^*)y \quad \text{for all } y \in \mathrm{Dom}(A_1), \tag{5.13}$$

from which one easily infers that both Λ_0 and Λ_1 are self-adjoint operators.

By using (5.8) and (5.9) one expresses the operators $Z_0 = A_0 + BK$, $\mathrm{Dom}(Z_0) = \mathrm{Dom}(A_0)$, and $Z_1 = A_1 - B^*K^*$, $\mathrm{Dom}(Z_0) = \mathrm{Dom}(A)$, in terms of Λ_0 and Λ_1. Then combining the expressions obtained with equality (2.24) from Corollary 2.9 we obtain formula (5.5). The similarity (5.5) means, in particular, that $\mathrm{spec}(L)$ is a Borel subset of $\mathbb{R}$. This completes the proof of part (i).

The J-orthogonality of the subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ is obvious since for any $x, y \in \mathfrak{H}$ of the form

$$x = x_0 \oplus Kx_0, \ x_0 \in \mathfrak{H}_0, \text{ and } y = K^*y_1 \oplus y_1, \ y_1 \in \mathfrak{H}_1, \tag{5.14}$$

we have $[x, y] = (Jx, y) = (x_0, K^*y_1) - (Kx_0, y_1) = 0$. Thus, the fist two assertions of part (ii) follow from Corollary 2.9 (ii). On the other hand, (5.14) yields $\|x\|^2 \leq (1 + \|K\|^2)\|x_0\|^2$ and $\|y\|^2 \leq (1 + \|K\|^2)\|y_1\|^2$, and, hence, combined with (5.11), it implies $[x, x] \geq \gamma\|x\|^2$ and $[y, y] \leq -\gamma\|y\|^2$ where $\gamma = (1 - \|K\|^2)(1 + \|K\|^2)^{-1} > 0$. This means that $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are maximal uniformly positive and maximal uniformly negative subspaces, respectively.

Now introduce the operators $T_0 = W_0(I - K^*K)^{-1/2}$ and $T_1 = W_1(I - KK^*)^{-1/2}$ where W_0 and W_1 are given in (2.28) assuming that $K' = K^*$. Taking into account (5.8) and (5.9), the identities (2.27) of Corollary 2.9 (ii) then imply

$$T_0^{-1}L|_{\mathfrak{H}_0'}T_0 = \Lambda_0 \quad \text{and} \quad T_1^{-1}L|_{\mathfrak{H}_1'}T_1 = \Lambda_1. \tag{5.15}$$

Clearly, $\text{Ran}\, T_0 = \mathfrak{H}_0'$, $\text{Ran}\, T_1 = \mathfrak{H}_1'$, $[T_0x_0, T_0y_0] = (x_0, y_0)$ for any $x_0, y_0 \in \mathfrak{H}_0$, and $[T_1x_1, T_1y_1] = -(x_1, y_1)$ for any $x_1, y_1 \in \mathfrak{H}_1$. This means that both $T_0 : \mathfrak{H}_0 \to \mathfrak{H}_0'$ and $T_1 : \mathfrak{H}_1 \to \mathfrak{H}_1'$ are $\mathfrak{K}$-unitary operators. Therefore, equalities (5.15) prove the remaining statement of part (ii).

The proof is complete. $\qquad\square$

Remark 5.3. By equalities (5.8) and (5.9) the self-adjoint operators Λ_0 and Λ_1 are similar to the operators

$$Z_0 = A_0 + BK, \ \text{Dom}(Z_0) = \text{Dom}(A_0), \ \text{and} \ Z_1 = A_1 - B^*K^*, \ \text{Dom}(Z_1) = \text{Dom}(A_1) \tag{5.16}$$

respectively, and, thus,

$$\text{spec}(\Lambda_0) = \text{spec}(Z_0) \ \text{and} \ \text{spec}(\Lambda_1) = \text{spec}(Z_1). \tag{5.17}$$

Notice that identities (5.12) and (5.13) imply that the operators Z_0 and Z_1 are self-adjoint on the corresponding Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ equipped with the new inner products $\langle f_0, g_0 \rangle_{\mathfrak{H}_0} = \big((I - K^*K)f_0, g_0\big)_{\mathfrak{H}_0}$ and $\langle f_1, g_1 \rangle_{\mathfrak{H}_1} = \big((I - KK^*)f_1, g_1\big)_{\mathfrak{H}_1}$, respectively.

Remark 5.4. The requirement $\|K\| < 1$ is sharp in the following sense: If there is no strictly contractive solution to the Riccati equation (5.4) then the operator matrix L may not be similar to a self-adjoint operator at all. This is clearly seen from the simple example below.

Example 5.5. Let $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}$. Set $A_0 = -\frac{d}{2}$, $A_1 = \frac{d}{2}$, and $B = b$ where b and d are positive numbers such that $b \geq \frac{d}{2}$. If $b > d/2$, the Riccati equation (5.4) has two solutions $X^{(1)} = \frac{d}{2b} + i\sqrt{\frac{d^2}{4b^2} - 1}$ and $X^{(2)} = \frac{d}{2b} - i\sqrt{\frac{d^2}{4b^2} - 1}$. Both $X^{(1)}$ and $X^{(2)}$ are not strictly contractive since $\|X^{(1)}\| = \|X^{(2)}\| = 1$. At the same time the spectrum of the matrix L consists of the two complex eigenvalues $\lambda_1 = i\sqrt{b^2 - \frac{d^2}{4}}$ and $\lambda_2 = -i\sqrt{b^2 - \frac{d^2}{4}}$. If $b = \frac{d}{2}$, the equation (5.4) has the only solution $X = 1$. In this case the spectrum of the matrix L is real (it consists of the only point zero) but one easily verifies by inspection that the only eigenvalue of L has a nontrivial Jordan chain and, thus, L is not diagonalizable. Therefore, in both cases $b > d/2$ and $b = d/2$ the matrix L cannot be made similar to a self-adjoint operator.

The next assertion represents a quite elementary corollary to Theorem 5.2.

Lemma 5.6. *Let the assumptions of Theorem 5.2 hold. Assume, in addition, that the spectra $\sigma_0' = \text{spec}(Z_0)$ and $\sigma_1' = \text{spec}(Z_1)$ of the operators Z_0 and Z_1 given by (5.16) are disjoint, that is, $\sigma_0' \cap \sigma_1' = \varnothing$. Then σ_0' and σ_1' are complementary spectral subsets of the block operator matrix L, $\text{spec}(L) = \sigma_0' \cup \sigma_1'$, and the graphs $\mathfrak{H}_0' = \mathscr{G}(K)$ and $\mathfrak{H}_1' = \mathscr{G}(K^*)$ are the spectral subspaces associated with the subsets σ_0' and σ_1', respectively.*

Proof. By the assumption the spectra $\operatorname{spec}(\Lambda_0) = \operatorname{spec}(Z_0) = \sigma_0'$ and $\operatorname{spec}(\Lambda_1) = \operatorname{spec}(Z_1) = \sigma_1'$ (see Remark 5.3) of the self-adjoint operators Λ_0 and Λ_1 given by (5.8), (5.9) are disjoint. Hence, the spectral projections $\mathsf{E}_\Lambda(\sigma_0')$ and $\mathsf{E}_\Lambda(\sigma_1')$ of the self-adjoint diagonal block operator matrix $\Lambda = \operatorname{diag}(\Lambda_0, \Lambda_1)$ associated with its spectral subsets σ_0' and σ_1' read simply as

$$\mathsf{E}_\Lambda(\sigma_0') = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{E}_\Lambda(\sigma_1') = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

By Theorem 5.2 (i) the operator L is similar to the operator Λ. This means that the similarity transforms $\mathsf{E}_L(\sigma_0') = T\mathsf{E}_\Lambda(\sigma_0')T^{-1}$ and $\mathsf{E}_L(\sigma_1') = T\mathsf{E}_\Lambda(\sigma_1')T^{-1}$ of the spectral projections $\mathsf{E}_\Lambda(\sigma_0')$ and $\mathsf{E}_\Lambda(\sigma_1')$ with T given by (5.6) represent the corresponding spectral projections of L. One verifies by inspections that $\mathsf{E}_L(\sigma_0') = Q_{\mathscr{G}(K)}$ and $\mathsf{E}_L(\sigma_1') = Q_{\mathscr{G}(K^*)}$ where $Q_{\mathscr{G}(K)}$ and $Q_{\mathscr{G}(K^*)}$ are given by (2.23) assuming that $K' = K^*$. That is, $\mathsf{E}_L(\sigma_0')$ and $\mathsf{E}_L(\sigma_1')$ are the (oblique) projections onto the graph subspaces $\mathscr{G}(K)$ and $\mathscr{G}(K^*)$, respectively, which completes the proof. □

Remark 5.7. The spectral projections $\mathsf{E}_L(\sigma_0') = Q_{\mathscr{G}(K)}$ and $\mathsf{E}_L(\sigma_1') = Q_{\mathscr{G}(K^*)}$ are orthogonal projections with respect to the Krein inner product (5.2).

From now on we will assume that the spectra of the entries A_0 and A_1 are disjoint and, thus, the sets $\sigma_0 = \operatorname{spec}(A_0)$ and $\sigma_1 = \operatorname{spec}(A_1)$ appear to be complementary disjoint spectral subsets of the total self-adjoint operator A. In such a case for any bounded perturbation V satisfying the bound $\|V\| < d/2$, $d = \operatorname{dist}(\sigma_0, \sigma_1)$, the spectrum of the perturbed operator $L = A + V$ consists of two disjoint subsets σ_0' and σ_1', lying in the closed $\|V\|$-neighborhoods $O_{\|V\|}(\sigma_0)$ and $O_{\|V\|}(\sigma_1)$ of the spectral sets $\sigma_0 = \operatorname{spec}(A_0)$ and $\sigma_1 = \operatorname{spec}(A_1)$, respectively. One can think of the sets σ_0' and σ_1' as the result of the perturbation of the corresponding spectral sets σ_0 and σ_1.

Provided that the perturbation V is J-symmetric and $\|V\| < d/2$, Theorem 5.8 below gives sufficient *a priori* conditions for the perturbed operator $L = A + V$ to remain similar to a self-adjoint operator. Hence, this theorem also gives sufficient conditions for the perturbed spectral sets σ_0' and σ_1' to remain on the real axis. Furthermore, the theorem presents the main result of the section giving for such V an *a priori* norm bound on variation of the spectral subspaces of A associated with the disjoint spectral subsets σ_0 and σ_1.

Theorem 5.8. *Assume Hypothesis 5.1 and choose one of the following:*

(i) *Assume (4.12) and set $\delta = \frac{2}{\pi}d$;*

(ii) *Assume (4.13) or (4.14) and set $\delta = d$.*

Also suppose that

$$\|V\| < \frac{\delta}{2}. \tag{5.18}$$

Then the spectrum of the operator L is purely real and consists of two disjoint components σ_0' and σ_1' such that

$$\sigma_0' \subset O_r\big(\operatorname{spec}(A_0)\big) \quad \text{and} \quad \sigma_1' \subset O_r\big(\operatorname{spec}(A_1)\big), \tag{5.19}$$

where

$$r = \|V\| \tanh\left(\frac{1}{2}\operatorname{arctanh}\frac{2\|V\|}{\delta}\right) < \|V\|.$$

Moreover, the operator L is similar to a self-adjoint operator and the same is true for the parts of L associated with the spectral subsets σ_0' and σ_1'. Furthermore, the following bound holds:

$$\tan\Theta_0 \leq \tanh\left(\frac{1}{2}\operatorname{arctanh}\frac{2\|V\|}{\delta}\right), \tag{5.20}$$

where $\Theta_0 = \Theta(\mathfrak{H}_0, \mathfrak{H}_0')$ *denotes the operator angle between the subspace* $\mathfrak{H}_0$ *and the spectral subspace* $\mathfrak{H}_0'$ *of L associated with the spectral subset* σ_0'. *Exactly the same bound holds for the operator angle* $\Theta_1 = \Theta(\mathfrak{H}_1, \mathfrak{H}_1')$ *between the subspace* $\mathfrak{H}_1$ *and the spectral subspace* $\mathfrak{H}_1'$ *of L associated with the spectral subset* σ_1'.

Proof. Under either assumption (i) or (ii) from Theorem 4.11 it follows that the Riccati equation (5.4) associated with the block operator matrix L has a solution $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$ that is unique in the ball $\mathscr{O}_{\delta/2\|B\|}(\mathfrak{H}_0, \mathfrak{H}_1)$ and satisfies the bound $\big($see formulas (4.18) and (4.20)$\big)$

$$\|K\| \leq \frac{\|V\|}{\frac{\delta}{2} + \sqrt{\frac{d^2}{4} - \|V\|^2}} = \tanh\left(\frac{1}{2}\operatorname{arctanh}\frac{2\|V\|}{\delta}\right). \tag{5.21}$$

Here we have taken into account that $\|B\| = \|V\|$. We refer to Remark 4.3 regarding the use of the hyperbolic tangent in (5.21).

Clearly, the bound (5.21) yields that the solution K is a strict contraction, $\|K\| < 1$. Then by Theorem 5.2 the block operator matrix L is similar to the self-adjoint operator Λ given by (5.7)–(5.9). Hence $\operatorname{spec}(L) \subset \mathbb{R}$ and $\operatorname{spec}(L) = \sigma_0' \cup \sigma_1'$ where $\sigma_0' = \operatorname{spec}(\Lambda_0)$ and $\sigma_1' = \operatorname{spec}(\Lambda_1)$. By Remark 5.3 we also have $\sigma_0' = \operatorname{spec}(Z_0)$ and $\sigma_1' = \operatorname{spec}(Z_1)$ where Z_0 and Z_1 are given by (5.16). Since $\|BK\| \leq \|V\|\|K\| \leq r$ and $\|B^*K^*\| \leq \|V\|\|K\| \leq r$, for the spectral sets $\sigma_0' = \operatorname{spec}(Z_0)$ and $\sigma_1' = \operatorname{spec}(Z_1)$ the inclusions (5.19) hold and these sets are disjoint, $\operatorname{dist}(\sigma_0', \sigma_1') \geq \delta - 2r > \delta - 2\|V\| > 0$. To prove the remaining statements of the theorem one only needs to apply Lemma 5.6 and then to notice that due to (2.14) we have $\|\tan\Theta_0\| = \|K\|$ and hence $\tan\Theta_0 \leq \|K\|$. Similarly, $\tan\Theta_1 \leq \|K^*\| = \|K\|$.

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.9. By the upper continuity of the spectrum, the inclusion $\operatorname{spec}(L) \subset \mathbb{R}$ also holds for $\|V\| = d/\pi$ in case (i) and for $\|V\| = d/2$ in case (ii).

Remark 5.10. In case (ii) the bounds (5.19) on the location of $\operatorname{spec}(L)$ and the bound (5.20) on the angle Θ_0 are optimal. The optimality of both (5.19) and (5.20) is seen from Examples 4.15 and 4.16 where one sets $c = b$.

Remark 5.11. Under condition (5.18) in both cases (i) and (ii) the perturbed spectral subspaces $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are mutually orthogonal with respect to the Krein space inner product (5.2) and, thus, $\mathfrak{K} = \mathfrak{H}_0'[+]\mathfrak{H}_1'$. These subspaces are maximal uniformly positive and maximal uniformly negative, respectively. The restrictions of L onto $\mathfrak{H}_0'$ and $\mathfrak{H}_1'$ are $\mathfrak{K}$-unitary equivalent to the self-adjoint operators Λ_0 and Λ_1 given by (5.8) and (5.9), respectively. By Theorem 5.2 (ii) all this follows from the fact that $\|K\| < 1$ which we established in the proof of Theorem 5.8.

Theorem 5.8 claims that the spectrum of the block operator matrix L is purely real whenever the off-diagonal J-self-adjoint perturbation V satisfies the bounds $\|V\| < d/2$ in case (i) or $\|V\| < d/\pi$ in case (ii). Recall that case (ii) corresponds to the general spectral situation where no constraints are imposed on the mutual positions of the spectra $\operatorname{spec}(A_0)$ and $\operatorname{spec}(A_1)$ except for the condition (4.12). Now we want to prove that, in fact, under the only condition (4.12) the spectrum of the operator L remains purely real even if $d/\pi \leq \|V\| < d/2$, at least in the case where the entries A_0 and A_1 are bounded. Our proof will be based on results from [32] and [49].

Theorem 5.12. *Assume Hypothesis 5.1. Assume, in addition, that both the entries* A_0 *and* A_1 *are bounded and such that* $\operatorname{dist}\big(\operatorname{spec}(A_0), \operatorname{spec}(A_1)\big) = d > 0$. *Also suppose that* $\|V\| < d/2$. *Then the spectrum of the block operator matrix L is real, that is,* $\operatorname{spec}(L) \subset \mathbb{R}$.

Proof. Under Hypothesis 4.8 and condition $\|V\| < d/2$ the inclusion $\mathrm{spec}(L) \subset \mathbb{R}$ has been already proven in Theorem 5.8 (ii). Thus, let us only consider the case that is not covered by Hypothesis 4.8. In this case, because of the separation condition $\mathrm{dist}\big(\mathrm{spec}(A_0),\mathrm{spec}(A_1)\big) = d$, the spectrum of A_0 consists of several (at least two) nonempty subsets isolated from each other at least by the distance $2d$. Denote these isolated spectral subsets of A_0 by $\sigma_0^{(i)}$, $i = 1, 2, \ldots, n_0$, $n_0 \geq 2$, assuming that they are numbered from left to right (i.e. $\sup \sigma_0^{(i)} < \inf \sigma_0^{(i+1)}$), the gap between $\sup \sigma_0^{(i)}$ and $\inf \sigma_0^{(i+1)}$ contains a nonempty subset of the spectrum of A_1, and $\bigcup_{i=1}^{n_0} \sigma_0^{(i)} = \mathrm{spec}(A_0)$. In exactly the same way, divide the spectrum of A_1 into the subsets $\sigma_1^{(j)}$, $j = 1, 2, \ldots, n_1$, $n_1 \geq 2$, so that $\bigcup_{j=1}^{n_1} \sigma_1^{(j)} = \mathrm{spec}(A_1)$, $\sup \sigma_1^{(j)} < \inf \sigma_1^{(j+1)}$, and $(\sup \sigma_1^{(j)}, \inf \sigma_1^{(j+1)}) \cap \mathrm{spec}(A_0) \neq \varnothing$. Denote by $\mathfrak{H}_0^{(i)}$, $i = 1, 2, \ldots, n_0$, and $\mathfrak{H}_1^{(j)}$, $j = 1, 2, \ldots, n_1$, the spectral subspaces of the operators A_0 and A_1 associated with the corresponding spectral subsets $\sigma_0^{(i)}$ and $\sigma_1^{(j)}$. Surely, $\oplus_{i=0}^{n_0} \mathfrak{H}_0^{(i)} = \mathfrak{H}_0$ and $\oplus_{i=1}^{n_1} \mathfrak{H}_1^{(i)} = \mathfrak{H}_1$.

Now take arbitrary unit vectors

$$e_0^{(i)} \in \mathfrak{H}_0^{(i)}, \ \|e_0^{(i)}\| = 1, \ i = 1, 2, \ldots, n_0, \ \text{and} \ e_1^{(j)} \in \mathfrak{H}_1^{(j)}, \ \|e_1^{(j)}\| = 1, \ j = 1, 2, \ldots, n_1, \quad (5.22)$$

and construct numerical matrices A_0, A_1, and B with the entries

$$\mathsf{A}_{0,ik} = (A_0 e_k^{(0)}, e_i^{(0)}), \quad \mathsf{A}_{1,jl} = (A_1 e_l^{(1)}, e_j^{(1)}), \quad \text{and} \quad \mathsf{B}_{ij} = (B e_j^{(1)}, e_i^{(0)}),$$

respectively. Consider the matrices A_0 and A_1 as operators resp. on $\widehat{\mathfrak{H}}_0 = \mathbb{C}^{n_0}$ and $\widehat{\mathfrak{H}}_1 = \mathbb{C}^{n_1}$, and B as an operator from $\widehat{\mathfrak{H}}_1$ to $\widehat{\mathfrak{H}}_0$. Out of the matrices A_0 and A_1 construct the block diagonal matrix $\mathsf{A} = \mathrm{diag}(\mathsf{A}_0, \mathsf{A}_1)$ and out of B and B^* the off-diagonal matrix $\mathsf{V} = \begin{pmatrix} 0 & \mathsf{B} \\ -\mathsf{B}^* & 0 \end{pmatrix}$. Both matrices A and V have dimension $n \times n$ where $n = n_0 + n_1$, and we consider them as operators on the n-dimensional space $\widehat{\mathfrak{H}} = \widehat{\mathfrak{H}}_0 \oplus \widehat{\mathfrak{H}}_1$.

Our nearest goal is to prove that the spectrum of the operator $\mathsf{L} = \mathsf{A} + \mathsf{V}$ is real. To this end, first, introduce the indefinite inner product

$$[x, y] = (x_0, y_0)_{\widehat{\mathfrak{H}}_0} - (x_1, y_1)_{\widehat{\mathfrak{H}}_1}, \quad x = x_0 \oplus x_1, y = y_0 \oplus y_1, x_0, y_0 \in \widehat{\mathfrak{H}}_0, x_1, y_1 \in \widehat{\mathfrak{H}}_1, \quad (5.23)$$

which turns the Hilbert space $\widehat{\mathfrak{H}}$ into a Krein (Pontrjagin) space. We denote the latter by $\widehat{\mathfrak{K}}$. The operator A is self-adjoint both on $\widehat{\mathfrak{H}}$ and $\widehat{\mathfrak{K}}$ while B only on $\widehat{\mathfrak{K}}$.

Then notice that for different i and k the vectors $e_i^{(0)}$ and $e_k^{(0)}$ belong to the different (and mutually orthogonal) spectral subspaces of A_0 and, hence, $\mathsf{A}_{0,ik} = \lambda_i^{(0)} \delta_{ik}$ where $\lambda_i^{(0)} = (A_0 e_i^{(0)}, e_i^{(0)})$ and δ_{ik} is the Kronecker's delta. Similarly, $\mathsf{A}_{1,jl} = \lambda_j^{(1)} \delta_{jl}$ where $\lambda_j^{(1)} = (A_1 e_j^{(0)}, e_j^{(0)})$. Clearly, both $\lambda_i^{(0)}$, $i = 1, 2, \ldots, n_0$, and $\lambda_j^{(1)}$, $j = 1, 2, \ldots, n_1$, are simple eigenvalues of A and, by construction of A, one has $\lambda_i^{(0)} \in \mathrm{conv}(\sigma_0^{(i)})$ and $\lambda_j^{(1)} \in \mathrm{conv}(\sigma_1^{(j)})$. This yields

$$\min_{i,k,\, i \neq k} |\lambda_i^{(0)} - \lambda_k^{(0)}| \geq 2d, \quad \min_{j,l,\, j \neq l} |\lambda_j^{(1)} - \lambda_l^{(1)}| \geq 2d, \quad \text{and} \quad \min_{i,j} |\lambda_i^{(0)} - \lambda_j^{(1)}| \geq d. \quad (5.24)$$

It is also obvious that, with respect to the inner product (5.23), the eigenvalues $\lambda_i^{(0)}$, $i = 1, 2, \ldots, n_0$, are of positive type, while the eigenvalues $\lambda_j^{(1)}$, $j = 1, 2, \ldots, n_1$, are of negative type.

Now to prove the inclusion $\mathrm{spec}(\mathsf{L}) \subset \mathbb{R}$ it only remains to observe that $\|\mathsf{V}\| \leq \|V\| < d/2$ and then to apply [32, Corollary 3.4] (cf. [16, Theorem 1.2]).

Since the inclusion $\mathrm{spec}(\mathsf{L}) \subset \mathbb{R}$ holds for any choice of the vectors (5.22), one then concludes that also $W^n(L) \subset \mathbb{R}$ where $W^n(L)$ denotes the block numerical range (see [49, Definition 2.1]) of the operator L with respect to the decomposition

$$\mathfrak{H} = \mathfrak{H}_0^{(1)} \oplus \ldots \oplus \mathfrak{H}_0^{(n_0)} \oplus \mathfrak{H}_1^{(1)} \oplus \ldots \oplus \mathfrak{H}_1^{(n_1)}. \tag{5.25}$$

By [49, Theorem 2.5] we have $\mathrm{spec}(L) \subset \overline{W^n(L)}$. Hence, $\mathrm{spec}(L) \subset \mathbb{R}$, which completes the proof. $\qquad\square$

Remark 5.13. By the upper continuity of the spectrum, under the hypothesis of Theorem 5.12 the spectrum of $L = A + V$ is real also for $\|V\| = d/2$ (cf. Remark 5.9).

Remark 5.14. Under the assumptions of Theorems 5.8 (ii) or 5.12 the requirement $\|V\| \leq d/2$ guaranteeing the inclusion $\mathrm{spec}(L) \subset \mathbb{R}$ is sharp. This is seen from Example 5.5 with $b > d/2$.

6. QUANTUM HARMONIC OSCILLATOR UNDER A $\mathscr{P}\mathscr{T}$-SYMMETRIC PERTURBATION

Let A be the Schrödinger operator for a one-dimensional quantum harmonic oscillator (see, e.g., [38, Chapter 12]). The corresponding Hilbert space is $\mathfrak{H} = L_2(\mathbb{R})$. Assuming that the units are chosen in such a way that $\hbar = m = \omega = 1$, the operator A reads

$$(Af)(x) = -\frac{1}{2}\frac{d^2}{dx^2}f(x) + \frac{1}{2}x^2 f(x), \quad \mathrm{Dom}(A) = \left\{ f \in W_2^2(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} dx\, x^4 |f(x)|^2 < \infty \right\}, \tag{6.1}$$

where $W_2^2(\mathbb{R})$ denotes the Sobolev space of those $L_2(\mathbb{R})$-functions that have their second derivatives in $L_2(\mathbb{R})$. The subspaces

$$\mathfrak{H}_0 = L_{2,\mathrm{even}}(\mathbb{R}) \ \text{ and } \ \mathfrak{H}_1 = L_{2,\mathrm{odd}}(\mathbb{R}) \tag{6.2}$$

of even and odd functions are the spectral subspaces of the (self-adjoint) operator A associated with the spectral subsets

$$\sigma_0 = \mathrm{spec}(A\big|_{\mathfrak{H}_0}) = \{n + 1/2 \mid n = 0, 2, 4, \ldots\} \ \text{ and } \ \sigma_1 = \mathrm{spec}(A\big|_{\mathfrak{H}_1}) = \{n + 1/2 \mid n = 1, 3, 5 \ldots\},$$

respectively (see, e.g., [44, p. 142]). Clearly, $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, the spectral sets σ_0 and σ_1 are disjoint,

$$d = \mathrm{dist}(\sigma_0, \sigma_1) = 1, \ \text{ and } \ \sigma_0 \cup \sigma_1 = \mathrm{spec}(A). \tag{6.3}$$

Let $\mathscr{P}$ be the parity operator on $L_2(\mathbb{R})$, $(\mathscr{P}f)(-x) = f(-x)$, and $\mathscr{T}$ the (antilinear) operator of complex conjugation, $(\mathscr{T}f)(x) = \overline{f(x)}$, $f \in L_2(\mathbb{R})$. An operator V on $L_2(\mathbb{R})$ is called $\mathscr{P}\mathscr{T}$-symmetric if it commutes with the product $\mathscr{P}\mathscr{T}$, that is, $\mathscr{P}\mathscr{T}V = V\mathscr{P}\mathscr{T}$ (see, e.g. [15, 16] and references therein).

In a particular case where the $\mathscr{P}\mathscr{T}$-symmetric potential V is an operator of multiplication by a function $V(\cdot)$ of $L_\infty(\mathbb{R})$, the following equality holds (see, e.g., [3]; cf. [32]):

$$\overline{V(x)} = V(-x) \ \text{ for a.e. } x \in \mathbb{R} \tag{6.4}$$

and hence

$$V^* = \mathscr{P}V\mathscr{P}. \tag{6.5}$$

Observe that the parity operator $\mathscr{P}$ represents nothing but the involution (1.3) associated with the complementary spectral subspaces (6.2) of the oscillator Hamiltonian (6.1). Therefore, the equality (6.5) implies that the $\mathscr{P}\mathscr{T}$-symmetric multiplication operator V is J-self-adjoint with respect the involution $J = \mathscr{P}$.

Any bounded complex-valued function V on $\mathbb{R}$ possessing the property (6.4) admits the representation

$$V(x) = a(x) + ib(x) \tag{6.6}$$

where both a and b are real-valued functions such that

$$a(-x) = a(x) \ \text{ and } \ b(-x) = -b(x) \ \text{ for any } x \in \mathbb{R}.$$

The terms $V_{\mathrm{diag}}(x) = a(x)$ and $V_{\mathrm{off}}(x) = ib(x)$ represent the corresponding parts of the multiplication operator V that are diagonal and off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, that is, with respect to the decomposition $L_2(\mathbb{R}) = L_{2,\mathrm{even}}(\mathbb{R}) \oplus L_{2,\mathrm{odd}}(\mathbb{R})$.

Now assume that V is an arbitrary bounded off-diagonal operator on $\mathfrak{H} = L_2(\mathbb{R})$ being J-self-adjoint with respect to the involution $J = \mathscr{P}$. One can choose in particular a $\mathscr{P}\mathscr{T}$-symmetric potential (6.6) with $a = 0$. By taking into account (6.3), from [15, Theorem 1.2] it follows that the spectrum of the perturbed oscillator Hamiltonian $L = A + V$, $\mathrm{Dom}(L) = \mathrm{Dom}(A)$, remains real (and discrete) whenever $\|V\| \leq 1/2$. If, in addition, the bound $\|V\| < 1/\pi$ is satisfied then one can tell much more: Under such a bound Theorem 5.8 (i) implies that L is similar to a self-adjoint operator. This theorem also gives bounds on the variation of the spectral subspaces (6.2):

$$\tan\Theta_j \leq \tanh\left(\frac{1}{2}\operatorname{arctanh}(\pi\|V\|)\right) < 1, \quad j = 0,1,$$

where $\Theta_j = \Theta(\mathfrak{H}_j, \mathfrak{H}'_j)$ stands for the operator angle between the subspace $\mathfrak{H}_j$ and the spectral subspace $\mathfrak{H}'_j$ of the perturbed oscillator Hamiltonian $L = A + V$ associated with the spectral subset $\sigma'_j = \mathrm{spec}(L) \cap O_{\|V\|}(\sigma_j)$, $j = 0,1$.

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